J. Fluid Mech. (1974), *vol.* 66, *part* 3, *pp.* 613–622
Printed in Great Britain

On the stability of extending films: a model for the film casting process

By Y. L. YEOW
Department of Mathematics, University of Singapore

(Received 25 September 1973 and in revised form 10 April 1974)

Isothermal Newtonian film flow is put forward as a simple model of the film casting process. Methods of linear hydrodynamic stability theory are applied to study the stability of the film flow. The relevant eigenvalue problems are formulated and solved numerically. Results are presented in the form of neutral-stability curves in the appropriate parameter space. For the case of two-dimensional disturbances stability results obtained here are compared with those of Pearson & Matovich (1969) and Gelder (1971) for the stability of isothermal Newtonian threadline flow.

1. Introduction

As a model of melt spinning of polymer fibres Matovich & Pearson (1969) studied the mechanics of molten threadlines undergoing steady non-uniform axially symmetric extension. Confining their attention to highly viscous systems, Pearson & Matovich (1969) investigated the spatial growth of small disturbances imposed at the die-exit end of the threadline. The imposed disturbances were assumed to have the general forms $a^*(x)\, e^{-i\omega t}$ and $v^*(x)\, e^{-i\omega t}$, where $a^*(x)$ and $v^*(x)$ represent the amplitudes of the radius and velocity disturbances; ω, assumed to be real, is the frequency of the imposed infinitesimal disturbances, x is the co-ordinate in the direction of flow with the die-exit at $x = 0$ and the wind-up mechanism at $x = l$ and t is time. As a measure of the sensitivity of the threadline flow to such disturbances, Pearson & Matovich obtained the amplification of the radius disturbance, i.e. $a^*(l)/a^*(0)$, as a function of the frequency and the overall extension of the threadline. The overall extension is defined as the ratio of the steady threadline velocity at $x = l$ to the velocity at $x = 0$. For the case where the velocity disturbance vanishes at both ends of the threadline the amplification curves of the radius disturbance exhibit singular points, i.e. the ratio becomes infinite for certain combinations of extension ratio and imposed frequency. These singular points were interpreted by Pearson & Matovich as possible unstable operating points of the melt spinning process. Gelder (1971) reformulated the disturbance equations of Pearson & Matovich as an eigenvalue problem. The parameters of the eigenvalue problem are ω and the overall extension of the threadline. ω is now permitted to be complex. By solving the eigenvalue problem, Gelder showed that at the singular points of Pearson & Matovich the eigenvalue problem has non-trivial neutrally stable solutions.

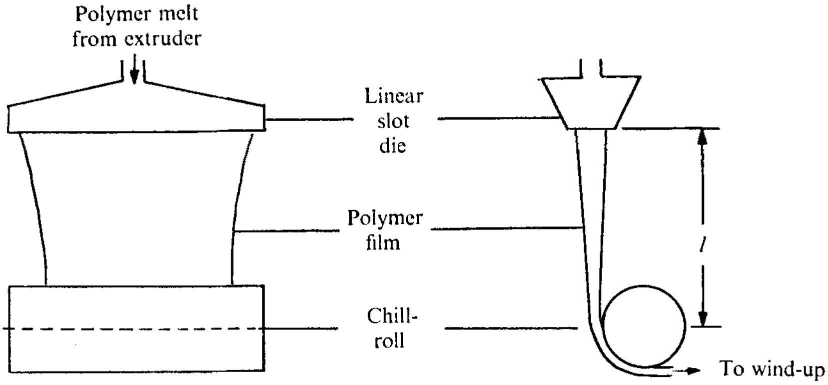

FIGURE 1. The film casting process.

The present work is concerned with the stability of the film casting process, which is shown schematically in figure 1. The equations governing the behaviour of infinitesimal disturbances are derived and the eigenvalue problem is set up and solved. Results are presented in the form of neutral-stability curves in the appropriate parameter space. If the disturbances are assumed to be uniform across the width of the film the eigenvalue problem of Gelder is re-obtained. However, with the introduction of disturbances which allow for variations across the width, the stability calculations for extending films become much more complicated.

2. Steady isothermal Newtonian film flow

A simple slow viscous model of the film casting process is introduced here. This forms the basis for the subsequent stability studies. This model, isothermal Newtonian film flow, considers only the melt drawing stage of the film casting process. Die swell has been analysed only recently (Zidan 1969; Richardson 1970) and will be ignored. The geometry of the model together with the coordinates and velocity components employed is shown in figure 2.

In film casting the polymer melt, which emerges continuously from the die, is allowed to fall vertically onto a chill-roll assembly. The linear speed of the film increases as it is drawn towards the chill-roll by the wind-up mechanism. Since, in practice, the width of the film is very large compared with its thickness and often much larger than the distance between the die and the chill-roll, the flow field can be considered to be effectively of infinite extent in the z direction. As a result, the flow can be regarded as two-dimensional and all steady flow quantities in the film flow model are assumed to be independent of z. The velocity is assumed to have no z component, i.e. $w = 0$. Experimental observations of the film casting process confirm the validity of these simplifying assumptions. Of course such a simple model fails to take into account the necking-in and thickening of the edges observed in practice. The mechanics of the flow near the edges are complex and have not been analysed. Until this has been done there is no means

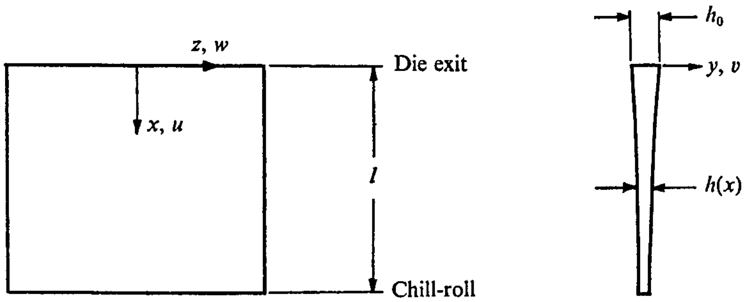

FIGURE 2. The film flow model.

of knowing the effects of the edges on the results of the stability analyses presented below.

As the polymer emerges from the die, the die-imposed velocity profile will rapidly decay away because of the great reduction in magnitude of the viscous forces acting on the surfaces of the fluid. Beyond the die-swell region it is reasonable to assume that the relaxation process is complete and the variation in the y direction of the x velocity is much less significant than its variation in the x direction. In the film flow model of the process it is therefore possible to approximate $u(x,y)$ by $u(x)$. Quantitative justification of this approximation would involve consideration of the magnitude of the surface forces, e.g. surface tension and aerodynamic drag (both of which are neglected in this study). The point is closely argued in Matovich (1966) and in Pearson & Petrie (1970).

With the above assumptions, as a consequence of the continuity equation for incompressible fluids, v can only depend linearly on y and can be written as

$$v = e_{yy}y, \tag{1}$$

$$e_{yy} = -du/dx. \tag{2}$$

It is obvious that e_{yy} is the yy component of the rate-of-strain tensor. To the degree of approximation made, the rate-of-strain tensor is

$$\begin{bmatrix} du/dx & 0 & 0 \\ 0 & -du/dx & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Assuming that the fluid is Newtonian with constant viscosity μ, the non-vanishing components of the stress tensor can be shown to be

$$t_{xx} = -p + 2\mu\, du/dx, \tag{3}$$

$$t_{yy} = -p - 2\mu\, du/dx, \tag{4}$$

$$t_{zz} = -p, \tag{5}$$

where p is the isotropic pressure. In most polymer film casting operations the viscosity of the melt is high and the velocity involved is small so the appropriate Reynolds number is small. Under these conditions viscous forces will dominate, and all but viscous and pressure terms in the momentum equations can be neglected. This then requires t_{yy} to remain constant in the flow field. If the ambient pressure is taken to be zero, this constant becomes zero. Equations (3) and (4) then give

$$p = -2\mu\, du/dx, \quad t_{xx} = 4\mu\, du/dx. \tag{6}$$

The integrated momentum balance in the x direction gives

$$ht_{xx} = f. \tag{7}$$

f is a constant which can be identified with the applied tension per unit width of the film (see appendix). The kinematic condition on the surface of the film requires, for steady flow,

$$\tfrac{1}{2}dh/dx = v/u \quad \text{at} \quad y = \pm\tfrac{1}{2}h.$$

This equation together with (1) and (2) gives

$$\frac{dh}{dx} = -\frac{h}{u}\frac{du}{dx}. \tag{8}$$

Q, the fixed volumetric flow rate per unit width of the film, is related to u and h by

$$Q = uh.$$

Solving (1)–(8) yields

$$u = u_0 \exp[fx/(4\mu Q)],$$
$$h = h_0 \exp[-fx/(4\mu Q)],$$
$$v = 2v_0 y \exp[fx/(4\mu Q)]/h_0,$$

where v_0 is defined by

$$v_0 = -u_0 h_0 f/(8\mu Q).$$

The physical interpretations of u_0, v_0 and h_0 are self-evident. The following dimensionless variables will now be introduced:

$$X = \frac{x}{l}, \quad Y = \frac{2y}{h_0}, \quad \overline{H} = \frac{h}{h_0}, \quad \overline{U} = \frac{u}{u_0}, \quad \overline{V} = \frac{v}{v_0},$$

together with the dimensionless parameter

$$\beta = fl/(4\mu Q).$$

l in the above expressions is the distance between the die and the chill-roll (see figure 1). Overbars are used to denote dimensionless properties of the steady flow. In terms of these dimensionless quantities, the steady film flow is described by

$$\overline{U} = e^{\beta X}, \quad \overline{H} = e^{-\beta X}, \quad \overline{V} = Y e^{\beta X}, \tag{9}$$

with $0 \leqslant X \leqslant 1$.

The single parameter β completely characterizes the steady film flow. β is the natural logarithm of the overall extension undergone by the film. For normal film casting operations β can be as large as 4·0 (Schenkel 1966, p. 335).

3. The disturbance equations

Infinitesimal disturbances are now introduced into the steady solution just obtained. The perturbed flow is assumed to be described by

$$\left.\begin{aligned} U &= e^{\beta X}[1+U^*(X)\,e^{i(\alpha Z-\Omega T)}],\\ V &= Y e^{\beta X}[1+V^*(X)\,e^{i(\alpha Z-\Omega T)}],\\ W &= e^{\beta X}[0+W^*(X)\,e^{i(\alpha Z-\Omega T)}],\\ H &= e^{-\beta X}[1+H^*(X)\,e^{i(\alpha Z-\Omega T)}]. \end{aligned}\right\} \tag{10}$$

The general notation of linear hydrodynamic stability analysis has been adopted here. The real parts of U^*, YV^* and W^* multiplied by the exponential factors represent the disturbances to the three velocity components. Similarly H^* represents the thickness disturbance. The dimensionless variables Z and T are defined by

$$Z = z/l, \quad T = tu_0/l.$$

α and Ω are respectively the dimensionless wavenumber and dimensionless frequency. For disturbances bounded in the Z direction α can only be real. Ω is generally complex; $\Omega = \Omega_r + i\Omega_i$. The disturbance velocity W^* has been made dimensionless with respect to u_0.

Substituting (10) into the dimensionless form of the continuity equation yields

$$\beta U^* + U^{*\prime} - \beta V^* + i\alpha W^* = 0. \tag{11}$$

In (11) and all subsequent equations relating the disturbance quantities, quadratic and higher-order terms in U^*, V^*, W^* and H^* have been suppressed. A prime denotes d/dx. Mass balance over an infinitesimal element of the film gives

$$\frac{\partial uh}{\partial x} + \frac{\partial wh}{\partial z} = -\frac{\partial h}{\partial t},$$

as shown in the appendix. In terms of infinitesimal disturbances this becomes

$$H^{*\prime} + U^{*\prime} + i\alpha W^* - i\Omega\, e^{-\beta X} H^* = 0. \tag{12}$$

If isothermal Newtonian behaviour with constant viscosity is assumed the non-zero components of the stress tensor in the perturbed flow can be shown to be

$$\begin{aligned} T_{xx} &= \overline{T}_{xx} + T^*_{xx} = 4d\overline{U}/dX + 2(\beta U^* + U^{*\prime} + \beta V^*)\, e^{i(\alpha Z-\Omega T)}\, e^{\beta X},\\ T_{zz} &= \overline{T}_{zz} + T^*_{zz} = 2(i\alpha W^* + \beta V^*)\, e^{i(\alpha Z-\Omega T)}\, e^{\beta X} + 2d\overline{U}/dX,\\ T_{xz} &= 0 + T^*_{xz} = (i\alpha U^* + \beta W^* + W^{*\prime})\, e^{i(\alpha Z-\Omega T)}\, e^{\beta X}. \end{aligned}$$

$\overline{T}_{xx}$ and $\overline{T}_{zz}$ are the dimensionless stress components in the steady flow. The stress tensor has been made dimensionless with respect to $\mu u_0/l$. Neglecting all but viscous terms, the momentum equations reduce to

$$\partial h t_{xx}/\partial x + \partial h t_{xz}/\partial z = 0,$$

$$\partial h t_{xz}/\partial z + \partial h t_{zz}/\partial z = 0$$

(see appendix). Converting these into dimensionless form and substituting in the expressions for the dimensionless stress tensor gives

$$4U^{*''}+4\beta U^{*'}-\alpha^2 U^*+3i\alpha W^{*'}+i\alpha\beta W^*+4\beta H^{*'} = 0, \tag{13}$$

$$3i\alpha U^{*'}+2i\alpha\beta U^*-4\alpha^2 W^*+\beta W^{*'}+W^{*''}+2i\alpha\beta H^* = 0. \tag{14}$$

To derive the last two equations, (11) was used to eliminate V^*.

Equations (12)–(14) form a fifth-order set of homogeneous simultaneous ordinary differential equations in U^*, W^* and H^*. There are two sets of boundary conditions for this differential system that are of interest. The first set is

$$\left.\begin{aligned} U^*(0) = W^*(0) = H^*(0) = 0,\\ U^*(1) = W^*(1) = 0, \end{aligned}\right\} \tag{15}$$

i.e. no velocity disturbance on the flow boundaries $X = 0$, 1 and no thickness disturbance at $X = 0$. These are the appropriate conditions for film casting operations in which the wind-up speed is held at fixed values. This case is the analogue of the constant-velocity spinning considered by Pearson & Matovich (1969) and Gelder (1971). The second set of boundary conditions is

$$\left.\begin{aligned} &U^*(0) = W^*(0) = H^*(0) = 0,\\ &W^*(1) = 0,\\ &2\beta U^*(1)+2U^{*'}(1)+2\beta H^*(1) = 0. \end{aligned}\right\} \tag{16}$$

This is the analogue of the constant-tension spinning analysed by Pearson & Matovich. The third equation in this set is obtained from the assumption that the applied tension remains constant in the perturbed flow.

4. The eigenvalue problem

Equations (12)–(14) together with the homogeneous boundary conditions (15) or (16) constitute an eigenvalue problem. The parameters of the problem are α, β and Ω. For a particular α and β there is a denumerably infinite number of Ω's for which the eigenvalue problem has non-trivial solutions. A direct numerical scheme has been developed to find these eigencombinations of the parameters α, β and Ω. The scheme is a simplification of the general scheme developed by Mack (1965) to deal with the stability of forced-flow compressible boundary layers. Briefly, the procedure involves finding a set of linearly independent solutions of (12)–(14). These solutions all satisfy the homogeneous boundary conditions at $X = 0$ of (15) or (16). They are combined so that all but one of the boundary conditions at $X = 1$ are satisfied. Finally the parameters are varied so that this last boundary condition is met as closely as desired. The linearly independent solutions are obtained by solving (12)–(14) as an initial-value problem (with appropriately chosen initial values). The actual integration of the differential system is performed by the standard fourth-order Runge–Kutta procedure. A simple linear search procedure is used to vary the combination of the parameters α, β and Ω. For numerical details see Yeow (1972). In general, for a given α and β, the scheme is applied to locate the first three of the infinitely

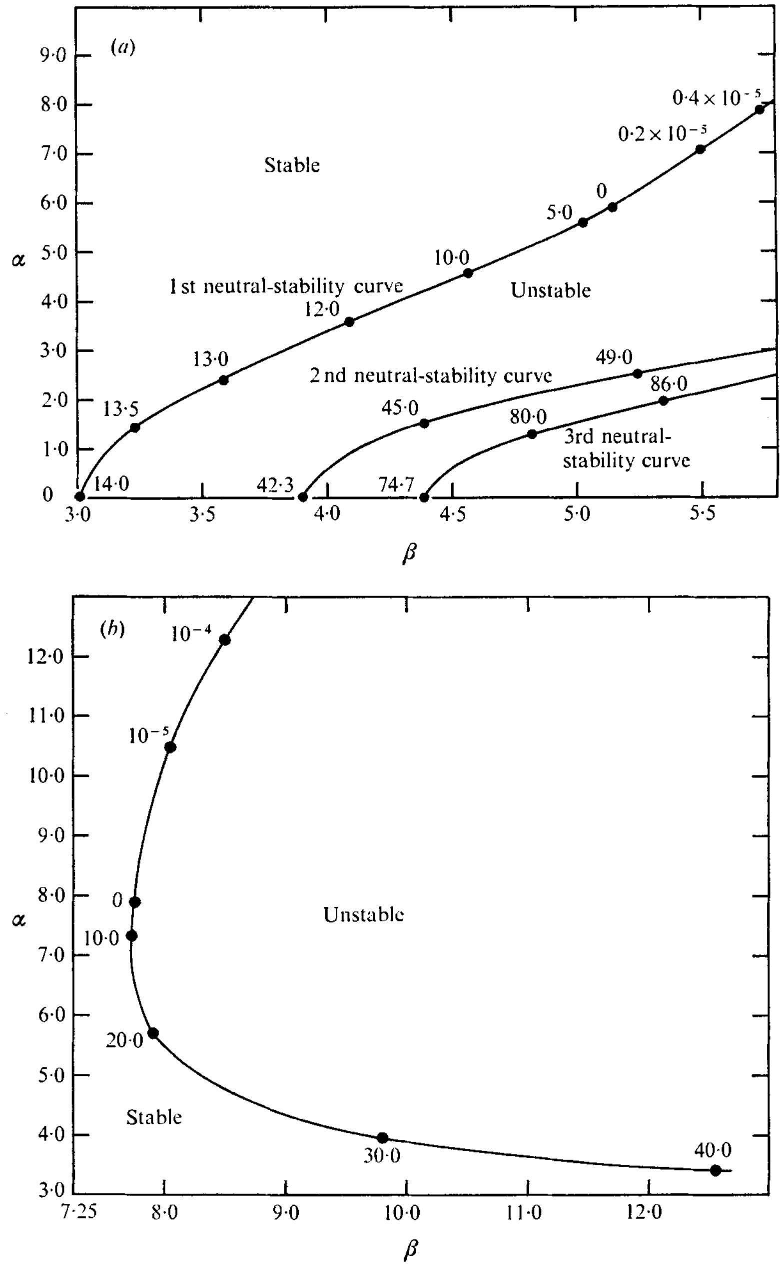

FIGURE 3. Neutral-stability curves for (*a*) constant-velocity and (*b*) constant-tension boundary conditions. Values of Ω_r indicated on curves.

many Ω's which together with α and β constitute eigencombinations of the parameters.

For a particular α and β each of these infinitely many Ω's corresponds to a mode of growth or decay of the disturbances. If all these Ω's have negative imaginary part, the flow, characterized by β, is stable to the disturbance characterized by the given α. If one or more of the Ω's have positive imaginary part the flow is considered unstable to this disturbance. The flow is only regarded as stable if it is stable for all α's. Neutral stability occurs when one of the Ω's is real. Inspection of (12)–(14) and the homogeneous boundary conditions shows that if (α, β, Ω) is an eigencombination then $(\alpha, \beta, -\bar{\Omega})$, where $\bar{\Omega}$ is the complex conjugate of Ω, is also an eigencombination. Clearly if (α, β, Ω) is stable/unstable then $(\alpha, \beta, -\bar{\Omega})$ is also stable/unstable. Therefore $\Omega_r < 0$ does not provide any new information about the system, and only $\Omega_r \geqslant 0$ need be considered.

The loci in the β, α plane of the neutrally stable eigencombinations of the parameters α, β and Ω are curves which separate the plane into stable and unstable regions. On these curves Ω_r varies continuously. The three neutral-stability curves corresponding to the first three modes of disturbance under constant-velocity boundary conditions are shown in figure 3(*a*). Some values of Ω_r are indicated on the curves. The regions of stability and instability are also indicated on the figure. Figure 3(*b*) gives the lowest neutral-stability curve for constant-tension boundary conditions. Some values of Ω_r are indicated on the curve. The regions of stability and instability are as shown in the figure.

5. Discussion

Consider the case of constant-velocity film casting. Figure 3(*a*) indicates that, as β is increased for a given α, there is a critical β beyond which the film flow becomes unstable. The smallest critical β is 3·006, corresponding to $\alpha = 0$. Since β, as defined, is the ratio of the applied tension to a typical viscous force, it can be concluded that, while viscous forces are stabilizing, the applied tension is destabilizing. In figure 3(*a*) all the neutral-stability curves intersect the β axis and the lowest β for transition from stability to instability occurs at $\alpha = 0$, i.e. there is an analogy with Squire's law for parallel shear flow, which states that two-dimensional disturbances (i.e. $\alpha = 0$) are more unstable than the three-dimensional ones. If α is set to zero in (12)–(15) it can be shown that the eigenvalue problem reduces to the one considered by Gelder. The intersections of the neutral-stability curves with the $\alpha = 0$ line are precisely the singular points of Pearson & Matovich and the neutral-stability points of Gelder for constant-velocity spinning. The stability of the Newtonian film flow to two-dimensional disturbances is governed by the same set of equations as that describing the stability of Newtonian threadline flow. This is however not necessarily the case if the material is assumed to be non-Newtonian.

A special feature of the first neutral-stability curve in figure 3(*a*) is worth noting. On this curve Ω_r decreases from 14·01 at $\alpha = 0$, $\beta = 3{\cdot}006$ to zero at $\alpha = 5{\cdot}80$, $\beta = 5{\cdot}14$. Beyond this point Ω_r increases again but very slowly. This means that at $\beta = 5{\cdot}14$ the film is capable of sustaining a steady periodic disturbance with wavenumber 5·80, i.e. the neutral mode at this point is a secondary flow. However this secondary flow is not realizable as for $\beta = 5{\cdot}14$ the flow is unstable to disturbances with wavenumber $\alpha < 5{\cdot}80$.

The first neutral-stability curve for boundary conditions of constant applied tension, figure 3(*b*), is quite different in shape from that of figure 3(*a*). In this case the neutral-stability curve does not intersect the β axis. By putting $\alpha = 0$ in (12)–(14) and boundary conditions (16) Yeow (1972) has shown that the eigenvalue problem has no non-trivial solution. This fact is consistent with the amplification curves of Pearson & Matovich, which show no singularity for spinning with constant applied tension. Under boundary conditions (16) the most unstable disturbance has a wavenumber of 6·9. This disturbance becomes unstable when $\beta = 7{\cdot}72$. The critical β is larger than the critical β for constant-velocity boundary conditions, and is unlikely to be attained in any actual process.

Results of the above analyses indicate that under constant-velocity boundary conditions the isothermal film flow is unstable to infinitesimal disturbances for $\beta > 3{\cdot}006$, i.e. overall extension $> 20{\cdot}1$. However overall extensions greater than 20·1 can be observed in normal film casting operations and are found to be stable. A number of factors neglected in the simple film flow model may contribute to the enhanced stability of the flow. In a series of papers Shah & Pearson (1972*a*, *b*, *c*) extended the work of Pearson & Matovich to study the effects of temperature–viscosity dependence and departure from Newtonian behaviour on the stability of fibre spinning. These authors concluded that these neglected factors can greatly enhance the stability of the threadline. Extension of the present analysis to include these factors can, in principle, be carried out; however, the mathematics and computation involved are likely to be much more complicated. This has not been attempted.

The author wishes to thank Prof. J. R. A. Pearson for supervising this work and the Lee Foundation, Singapore, for providing financial support. The major part of the work was carried out in the Department of Chemical Engineering, University of Cambridge.

Appendix. Derivation of the integrated force and mass balance equations

If all but viscous terms are neglected, the integral form of the equation of motion for a fluid reduces to

$$\oint_S t_{ij} n_j \, dS = 0.$$

The surface integral is taken over an arbitrary closed surface S in the fluid. Consider the infinitesimal film element lying between x and $x+dx$ and z and $z+dz$. In the case of steady flow, the viscous stress tensor t_{ij} is non-vanishing over the two areas normal to the direction of flow and zero elsewhere. The above equation reduces to

$$dht_{ij}/dx = 0$$

and hence (7). In the case of the perturbed flow, t_{ij} vanishes only on the two free surfaces. The x component of this equation reduces to

$$\partial ht_{xx}/\partial x + \partial ht_{xz}/\partial z = 0$$

and the z component to

$$\partial ht_{xz}/\partial x + \partial ht_{zz}/\partial z = 0.$$

In integral form the continuity equation for an incompressible fluid is

$$\int_S v_i n_i dS = \frac{\partial}{\partial t}\int_\Gamma d\Gamma,$$

where Γ is the volume enclosed by the closed surface S. Consider again the film element lying between x and $x+dx$, and z and $z+dz$. On the free surfaces the velocity has no normal component and hence there is no contribution to the surface integral. In this case the continuity equation yields

$$\frac{\partial uh}{\partial x}+\frac{\partial wh}{\partial z} = -\frac{\partial h}{\partial t}.$$

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